Indian Statistical Institute
End Semestral Examination- B.Math-I
Algebra-II
2nd May 2011

Time : 3 hours Max. Marks : 100

Answer question **1** and any **five** from the rest.

(1) Prove the following statements.
(a) Similar matrices have the same eigen values.
(b) A matrix is nilpotent if and only if all eigenvalues are zeroes.
(c) Every matrix A such that $A^2 = A$ is similar to a diagonal matrix.
(d) Determinant of a hermitian matrix is real.
(e) Every conjugacy class in the unitary group contains a diagonal matrix.

5+5+5+5+5

(2) Let A be the matrix $\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$.
(a) Show that the characteristic polynomial for A is $x^2(x-1)^2$.
(b) What is the minimal polynomial for A?
(c) Is A diagonalizable over $\mathbb{C}$? Give reasons.

5+5+5

(3) Let V be a finite dimensional vector space over $\mathcal{F}$ and let T be a linear operator on V. Then show that T is triangularizable if and only if the minimal polynomial for T is a product of linear polynomials over $\mathcal{F}$.

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(4) (a) Let T be a linear operator on a n dimensional vectror space V, and suppose that T has n distinct eigenvalues. Prove that T is diagonalizable.
(b)Let T be a linear operator on a space of dimension 2. Assume that the characteristic polynomial of T is $(x-a)^2$. Prove that there is a basis of V such that the matrix of T has one of the two forms $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$.

8+7

(5) a) Prove that an $n \times n$ matrix A over the real numbers is a positive definite symmetric matrix if and only if $A = P^t P$ for some $P \in GL_n(\mathbb{R})$.
b) Let V be a real vector space of finite dimension n with positive definite symmetric bilinear form $\langle , \rangle$. Show that the group of isometries of $(V, \langle , \rangle)$ (linear operators $T : V \longrightarrow V$ such that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in V$) is isomorphic to the group $O_n(\mathbb{R})$ of invertible $n \times n$ real matrices P such that $PP^t = I_n$.

10+5

(6) State and prove Sylvester's law for a symmetric form on a finite dimensional real vector space.

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P.T.O.

(7) (a) Prove that if T is a hermitian operator on a hermitian vector space V, then there is an orthonormal basis of V consisting of eigen vectors of T.
(b) State the matrix analogue of the above statement and show that the two are equivalent.

8+7

(8) (a) State spectral theorem for normal matrices.
(b) Let A be a normal matrix. Prove that A is hermitian if and only if all eigen values of A are real.
(c) Let A be a normal matrix. Prove that A is unitary if and only if every eigen value has absolute value 1.

3+6+6